Analytical Approximations to Type III Separation Thresholds in Poincaré Halfmaps

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The curves Ω_k, marking the frequency thresholds for type III separation in the parameter space of Poincaré halfmaps, are investigated further analytically. In the limit of great values of their argument, the relaxation constant ϱ, all frequency functions Ω_k are shown to grow algebraically – each with the same exponent being $\frac{2}{3}$. Furthermore, a perturbation expansion is presented that yields good results already at a level of approximation where the calculations can be performed analytically.

1. Introduction

Piecewise-linear continuous dynamical systems proved helpful in analyzing the dynamical behavior of different problems like hormonal regulation [1, 2], nerve conduction equations [3], compliant off-shore structures [4], or electronic circuits [5], to mention just a few. For appropriate values of the system parameters, all these examples exhibit chaotic behavior, which can be understood in terms of mechanisms that separate trajectories with adjacent initial conditions. Such a separating mechanism has to appear at least for one parital dynamics of the system; a recurrent behavior, however, which yields an attractor, requires the interaction of all the partial dynamics involved. For the simplest cases, with only two regions in state space, the notion of Poincaré halfmaps [6] turned out to be a fruitful concept. These maps (denoted here as $\bar{P}$ and $\tilde{P}$) are defined in the plane S, which separates the two regions, called $\bar{T}$ and $\tilde{T}$ [7].

When a saddle-focus type dynamics in three space (with an actual fixed point [8] and negative real eigenvalue) generates a halfmap, the mapping behaves discontinuously along a branch cut, the critical spiral γ, being from the domain of the map [6, 7]. The latter property of halfmaps exhibits a separating mechanism, which is due to trajectories that touch the plane S. They lie in between those orbits that exit the region close to the point of contact of the touching trajectory and those which perform almost one more turn around the real eigenvector of the saddle-focus.

This separation is called "type I" for values of the canonical parameters below the curve $\omega^2 = \varrho + 1$, and is denoted "type II" above the latter threshold [7]. For a properly chosen second halfmap both types of separation give rise to a positive Lyapunov exponent [9] and hence chaotic solutions.

As we tune through the canonical parameters [10] of a saddle-focus (specifically, increase the frequency ω for a fixed value of the relaxation constant ϱ), beyond a threshold value, the critical spiral becomes object to a *selection rule* which cuts out segments from its naively found ancestor, the curve $\tilde{\gamma}$. That is, its domain in the (parametric) τ-representation looses connectivity. While almost all points of γ are concerned to orbits which are touching the separating plane S once, a "restart point" of the spiral (s_k) marks the entry (into $\bar{T}$) and a "break point" of γ (b_k) is the first point of contact of a "doubly touching trajectory". This yields a third (stronger) kind of separation, called "type III". Since no trajectory can touch the plane S more than twice, this is the most complicated behavior possible for a single halfmap [7].

The sets of parameter values, where doubly touching orbits first come into existence, form the curve Ω_1 in the ϱ-ω-plane of the canonical parameter space [7]. Beyond Ω_1 a countable infinity of curves Ω_k ($k = 2, 3, \ldots$) exists. There the kth pair of break and restart points of γ freshly appears; together with a doubly touching trajectory, which performes k turns around the real eigenvector of the saddle-focus in between its two points of contact (b_k and c_k) with S.

An equivalent description of this situation can also be given in terms of the critical curve φ, which is formed by the union of exit points of trajectories that enter the region $\bar{T}$ in a point of γ, i.e., φ is from the

range of the halfmap. In the latter case the parameters on Ω_k mark a situation where φ reaches the line W (separating the entry from the exit points of the region $\bar{T}$) for the kth time and therefore a new branch of the critical curve has to start in s'_k, the exit point of the new born doubly touching orbit. This gives φ a fanlike shape, which again reflects a selection rule [11].

Although the two conditions for obtaining Ω_k formally look rather different, as far as the canonical parameters and the mapping times are concerned, both procedures have to yield the same results – since both describe the appearance of doubly touching orbits. The formulas in the "φ-representation" are, however, considerably easier to handle. Thus, in the present paper, we choose the latter approach to determine the behavior of the Ω_k.

In order to obtain the exact threshold curves, a system of transcendental implicit equations has to be solved numerically. This yields a sequence of frequencies $\Omega_k(\varrho)$ for each value of ϱ. Here an analytical approximation is presented. To lowest order the result comes out to be rather simple, having the form of an algebraic growth law with the same exponent, being $\frac{2}{3}$, for all k. The next order of approximation is shown to converge towards the exact solutions for great values of ϱ. These results allow further insight into the structure of bifurcations, where a chaotic attractor changes its character.

2. A Short Review of some Results Concerning the Critical Curve φ

The canonical equation of motion [10] for the dynamics to be treated here reads

$$\frac{dx}{dt} = \varrho x \qquad (1\,a)$$

and

$$\frac{dy}{dt} = -(1 + i\omega)y \qquad (1\,b)$$

with x, ϱ, $\omega \in \mathbb{R}$ ($\varrho > 1$, $\omega > 0$) and $y(= \eta + i\xi) \in \mathbb{C}$ This dynamics is assumed to be active inside the region $\bar{T}$ and induces the map $\bar{P}^{-1}$ with domain S^+ and range S^- [10]. From now on the latter halfmap will be considered alone.

As was explained already in [11], the present dynamics is the time reverse of the one treated in [7] and [10]. Thus, whenever we talk about "trajectories",

"entry" or "exit points", we adopt the opposite time direction, in order be consistent with the previous papers. Thus, although the present equation of motion yields a growth of the system in the x-direction, ϱ will be denoted as a "relaxation constant". This will help us to avoid clutter and confusion – all terms recur to the original dynamics.

In the above coordinates the separating plane S, where the overall dynamics switches its behavior, is characterized by

$$x + 2\eta = x_0, \qquad (2)$$

where the factor "2" is due to the metric of the state space $\mathbb{R} \times \mathbb{C}$. The parameter x_0 determines the distance of the fixed point from the separating plane. It introduces a "natural length scale" in state space, which essentially performs the anchor of nonlinearity in our theory. However, since the quantity x_0 will not appear in the parameter equations, the results described are valid at all length scales given by x_0. The η-coordinates on S are obviously a function of x alone:

$$\eta(x) = \frac{x_0 - x}{2}. \qquad (2\,a)$$

By the direction of the intersecting trajectories, the separating plane is in a natural manner divided into halves, S^+ and S^-, being the union of exit and entry points, respectively, of the region $\bar{T}$. Their boundary are the non-transversal points of S, which form the straight line W, that can be represented by the linear equation

$$\xi_W(x) = \frac{x_0 - (\varrho + 1)x}{2\omega}, \qquad (3)$$

see [10]. The point $(x_c, 0)^T$, with

$$x_c = \frac{x_0}{\varrho + 1}, \qquad (3\,a)$$

where W crosses the x-axis, plays an important role in the theory of Poincaré halfmaps [10].

Let us now turn to the intersection (with S) of trajectories which touch the separating plane in a point of W. From the formalism for calculating critical curves, as it was developed in [8, 12], the curve

$$\tilde{\varphi}(\tau) = (x_\tau^{(\varphi)}, \xi_\tau^{(\varphi)}(x_\tau^{(\varphi)}))^T \qquad (4)$$

is obtained in S (since the η-components are known from (2a), they are again omitted). Hereby the two functions that appear in the components of $\tilde{\varphi}$ read in

the parametric τ-representation

$$x_\tau^{(\varphi)} = x_0 \frac{z^{(\varphi)}(\tau)}{n^{(\varphi)}(\tau)} \qquad (4\,a)$$

and

$$\xi_\tau^{(\varphi)}(x) = \frac{-e^{(\varrho+1)\tau} + \cos \omega \tau}{2 \sin \omega \tau} x$$
$$+ \frac{e^\tau - \cos \omega \tau}{2 \sin \omega \tau} x_0, \qquad (4\,b)$$

respectively. In the present case, the important (numerator and denominator) functions $z^{(\varphi)}$ and $n^{(\varphi)}$ turn out to be

$$z^{(\varphi)}(\tau) = \omega - e^\tau(\omega \cos \omega \tau - \sin \omega \tau) \qquad (5\,a)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5\,b)$

$$n^{(\varphi)}(\tau) = \omega - e^{(\varrho+1)\tau}(\omega \cos \omega \tau - (\varrho+1)\sin \omega \tau),$$

cf. [11]. These functions contain already the whole information that is required to construct the curves Ω_k.

As we already saw with γ, the critical curve φ too has to be extracted from the above $\tilde{\varphi}$ by applying a specific selection rule [11]:

(S φ). *Only the first appearance of a value $x_S^{(\varphi)}(\tau)$ for $\tau \in \mathbb{R}^+$ is to be considered as physically meaningful. That is, $\tilde{\varphi}(\tau)$ is a point of φ iff*

$$\tau = \inf\{\tau' \,|\, x_S^{(\varphi)}(\tau') = x_S^{(\varphi)}(\tau)\}. \qquad (6)$$

Here $x_S^{(\varphi)}(\tau)(:= x_\tau^{(\varphi)} \cdot e^{\varrho \tau})$ is the x-coordinate of the point of contact that a trajectory, which crosses S in $\tilde{\varphi}(\tau)$, has with the separating plane. This rule removes nonunique inverses (shadow or ghost solutions [13]) of the halfmap, which arise when apparently different intersection points lead to the same point of contact (compare Figs. 1 and 5 of [11] for the action of $S\,\varphi$; see also [14] for the discussion of selection rules and ghost solutions in three-region piecewise-linear dynamical systems).

Adjacent to the entry point of a doubly touching trajectory there are initial values of orbits that touch S close to the second point of contact (c_k), however, after having left $\bar{T}$ already near the first point of contact (b_k). Thus the onset of a selection coincides with crossing a curve Ω_k in parameter space towards higher frequencies. (For the sake of brevity, we shall not distinguish between a point $(\varrho, \Omega_k(\varrho))^T$ on the curve and the value of its frequency coordinate.) Originally, these curves were constructed from the properties of the critical spiral γ. At a point of Ω_k the first two derivatives of the function $x_s^{(\gamma)}(:= x_\tau^{(\gamma)} \cdot e^{-\varrho \tau})$ have to

vanish for one specific value of τ [7]. One major result of [11] was that a collision of a pole and a zero of the function $x_S^{(\varphi)}$, which results in the finite value

$$x_W \left(= x_0 \frac{\omega^2 + 1}{(\varrho+1)^2 + \omega^2}, \text{ see [7]} \right),$$

is equivalent to the above condition. The latter requirement is fulfilled at simultaneous zeros of $z^{(\varphi)}$ and $n^{(\varphi)}$, as can be seen immediately. This approach yields rather concise expressions (compared to (34) and (36) of [7]) and hence requires considerably less numerical effort than the old procedure. In the present case we have to solve the following system of implicit parameter equations:

$$z^{(\varphi)}(\tau_k^c; \varrho, \Omega_k) = 0 \qquad (7\,a)$$

and

$$n^{(\varphi)}(\tau_k^c; \varrho, \Omega_k) = 0. \qquad (7\,b)$$

For an arbitrarily fixed value of the relaxation constant ϱ, a solution of (7) is the "mapping time" [10] $\tau_k^c(\varrho)$ and the frequency $\Omega_k(\varrho)$. Although the mapping time appears just as a parameter of representation, it gains importance by being the time, which the system needs to move from the entry or exit point, respectively, to the point of contact with S. Thus the product $\Omega_k \tau_k^c$ is a physically meaningful and measurable quantity, a phase, which is even independent of the actual gauging process, that was employed to obtain the canonical form of the equation of motion. It is in fact the only gauge-invariant quantity in the theory.

Another important result of [11] was the fact that the phase functions $\Omega_k \tau_k^c$ are almost constant for all values of ϱ and that they approach integer multiples of 2π, as ϱ grows. Here we are going to present some interesting consequences of this result. Specifically the limiting growth law of the curves Ω_k will be demonstrated explicitly.

3 The Limiting Growth Law for the Curves Ω_k

In order to obtain detailed information on the behavior of the Ω_k for great values of ϱ, we first need a little Lemma that gives us an upper bound for the speed of growth of these functions.

Lemma. *For $\varrho \to \infty$ the curves Ω_k can be estimated by*

$$\Omega_k(\varrho) < \mu_k \varrho^{1-\varepsilon}, \qquad (8)$$

with $\mu_k > 0$ and $0 < \varepsilon < 1$. Furthermore,

$$\varrho \cdot \tau_k^c(\varrho) \xrightarrow{\varrho \to \infty} \infty \qquad (9)$$

holds true.

The proof can be found in the *Appendix*.

This result allows us to estimate several expansions that will appear in the following. Eventually, making use of the "almost constant phase property" from [11], we are going to formulate two ($\varrho \to \infty$)-versions of (7) being the beginning of a perturbation expansion and which can be solved analytically. The solutions of these equations are denoted by ${}^j_i\bar{\Omega}_k$. Here the bar indicates the limiting property, the lower left index "i" shows the order of approximation and the upper left one "j" gives the iteration order (if there is any). For the τ_k^c a similar convention applies.

As a first step we make use of the fact that the phases where (7) is fulfilled are almost constant. Consequently their deviation functions (or, equivalently, shift functions) are defined as

$$\delta_k(\varrho) := \bar{\Omega}_k(\varrho) \cdot \bar{\tau}_k^c(\varrho) - \Pi_k, \tag{10}$$

where the $\Pi_k := 2 k \pi$ are "ideal phases", i.e., the limiting values for $\varrho \to \infty$. All δ_k go towards zero for great values of ϱ, as can be seen from (A 1) in the Appendix. Hence it appears reasonable to expand $z^{(\varphi)}$ and $n^{(\varphi)}$ both about the Π_k in terms of δ_k. Equation (10) hereby provides a nonlinear transformation which switches from (Ω_k, τ_k^c)- to (Ω_k, δ_k)-coordinates.

Since ϱ does not appear explicitly in (5 a), the system of implicit equations is of "lower triangular" type and can hence be solved iteratively. In the first step, $z^{(\varphi)}$ is transformed via (10) and then expanded to terms containing δ_k quadratically. If only the fastest growing terms are considered, the zeros of the expansion turn out to fulfill

$${}_1\bar{\Omega}_k(\varrho)\, \delta_k(\varrho)^2 = 2\, \Pi_k. \tag{11}$$

This is already an important result, it yields a connection between the phase shift and the frequency as they appear along the curves Ω_k in parameter space. At a first sight, the latter relation is correct to order δ_k^0. However, since the first neglected term of the expansion is $\Pi_k \delta_k^2$, (11) is indeed correct to order δ_k. Thus it is a proper tool for an investigation of the limiting behavior of Ω_k.

After having solved (7 a) to the desired level of accuracy, let us now turn to the zeros of $n^{(\varphi)}$. Here, due to our Lemma, the exponential $e^{(\varrho+1)\tau_k^c}$ grows much faster than Ω_k, hence the zeros of the rightmost expression (in brackets) of (5 b) are our next objective. For a result of lowest order it suffices to expand to terms containing δ_k. This is again justified by (8) of the *Lemma*: $\varrho\, \delta_k$ grows faster than $\Omega_k \delta_k^2$, and it does so at least by a factor of order δ_k^{-1}. Hence all terms that

grow like $\varrho\, \delta_k$ or faster have been taken into consideration. This yields as a first approximation on (7 b)

$$0 = {}_{-1}\bar{\Omega}_k(\varrho) - (\varrho+1)\,\delta_k(\varrho). \tag{12}$$

Upon inserting now (11),

$${}_{-1}\bar{\Omega}_k(\varrho) = \sqrt[3]{2\, \Pi_k(\varrho+1)^2} \tag{13}$$

is immediately obtained. Equation (13) is our major result: We have evolved an *explicit expression* for the limiting behavior of the Ω_k. It shows that all these curves essentially differ just by a factor $\sqrt[3]{2\, \Pi_k}$, while they altogether grow approximately like $\varrho^{2/3}$.

Let us now compare the latter family of functions to the the numerically found "true" Ω_k. In computer experiments (with $\varrho \approx 50\,000$), we find the relative error $\dfrac{\Omega_k - {}_{-1}\bar{\Omega}_k}{\Omega_k}$ to approach zero while the absolute error, the numerator of the latter expression, tends toward a finite constant value. This reflects the fact that (13), although it contains only terms which grow like δ_k^{-2}, is correct to order δ_k^{-1} (as indicated). The reason for this behavior is the absence of terms, growing similar to δ^{-1}, from the expansion of $n^{(\varphi)}$. Hence a remaining constant error appears unavoidable at the present level of accuracy. That is, our lowest order result yields a limiting *growth law*.

Figure 1 shows the first five numerically found exact curves Ω_k compared to the growth law behavior ${}_{-1}\bar{\Omega}_k$
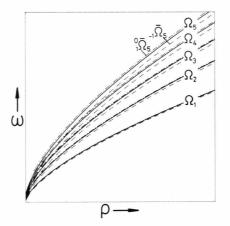

Fig. 1. The first five curves Ω_k (solid lines) each compared to the two explicit approximations ${}_{-1}\bar{\Omega}_k$ (— — —) and ${}^0_1\bar{\Omega}_k$ (— · —). While for the first approximation (to order δ_k^{-1}), the relative error decreases, leaving an almost constant absolute error, the curves to order δ_k^1 remove the remaining deviation for big enough values of ϱ. Parameters: $x_0 = 1$. Axes: $\varrho = 1 \ldots 1000$, $\omega = 0 \ldots 400$.

of (13) and a higher order approximation to be discussed in Section 4. At least for k up to three, (13) yields already a good approximation. The grade of accuracy, however, gets worse as the value of k increases.

4. Corrections to the Growth Law

Let us now improve upon (13). In contrast to this "final result", the link between the frequency and the phase deviation function from (11) is already correct to order δ_k. Thus, unlike (12), it needs not be refined further for the present purpose, and we can immediately turn again to (7b). Expanding $n^{(\varphi)}$ to order δ_k^0 (i.e., taking the first two terms of the sine and cosine power series) yields a condition for $_1\bar{\Omega}_k$:

$$0 = (1 - \overbrace{e^{-(\varrho+1)(\Pi_k + \delta_k(\varrho))/_1\Omega_k(\varrho)}}^{=:E(_1\bar{\Omega}_k)}) \,_1\bar{\Omega}_k(\varrho) - \overbrace{(\varrho+1)\delta_k(\varrho)}^{\mathcal{O}(|\delta_k^{-2}|)}$$
$$\underbrace{- \,_1\bar{\Omega}_k(\varrho)\frac{\delta_k(\varrho)^2}{2} + (\varrho+1)\frac{\delta_k(\varrho)^3}{6}}_{\mathcal{O}(|\delta_k^0|)}. \tag{14}$$

In order to treat the first term, ω, of (5b) properly, we in fact expanded the function $n^{(\varphi)} \cdot e^{-(\varrho+1)\tau}$, which obviously has the same zeros as $n^{(\varphi)}$ itself. After some algebra, utilizing (11), eventually

$$0 = \overbrace{[1 - E(_1\bar{\Omega}_k)]^2 \cdot {}_1\bar{\Omega}_k(\varrho)^3 - 2\Pi_k[1 - E(_1\bar{\Omega}_k)] \cdot {}_1\bar{\Omega}_k(\varrho)^2 + \Pi_k^2 \cdot {}_1\bar{\Omega}_k(\varrho)}^{=:P(_1\bar{\Omega}_k)}$$
$$\underbrace{- 2\Pi_k(\varrho+1)^2 \cdot \left[1 - \frac{2\Pi_k}{3 \cdot {}_1\bar{\Omega}_k(\varrho)} + \frac{\Pi_k^2}{9 \cdot {}_1\bar{\Omega}_k(\varrho)^2}\right]}_{=:C_0(_1\bar{\Omega}_k)} \tag{15}$$

is obtained.

In order to handle this expression, let us compare the speed of growth for the different constituents of (15). First the function E: It behaves like $\exp(-\delta_k^{-1}) \approx \exp(-\sqrt[3]{\varrho})$ and thus yields only a minor contribution compared to unity from which it is subtracted. Since it decreases exponentially, we shall not take it into consideration when discussing the other, algebraically behaving terms. Next, the $_1\bar{\Omega}_k^3$-term and the first one in C_0 both grow like δ_k^{-6}. If only they are considered, obviously (13) has to reappear, with a correction due to the exponential E. The second term in P and the middle one of the function C_0 grow to

order δ_k^{-4}, respectively. They yield the first correction to the growth law. The two remaining terms go like δ_k^{-2}. In consequence their contributions are the smallest.

Although in (15) only terms of order δ_k^{-2} and faster growing expressions appear, it is correct to order δ_k^1, as is indicated. This results from the fact that the latter equation was obtained from (14) by taking squares and multiplying it by a factor $_1\bar{\Omega}_k$, which introduces a term of order δ_k^{-2}. Another order of δ_k is gained by an argument mentioned already: When the sine and cosine from $n^{(\varphi)}$ are expanded in a power series, all terms that appear behave like even powers of δ_k.

After having compared the different speeds of growth, it appears reasonable to substitute our above result for the type III thresholds, that is $_{-1}\bar{\Omega}_k$, for $_1\bar{\Omega}_k$ in the functions E and C_0. Then (15) becomes a cubic equation (with coefficients $C_0(_{-1}\bar{\Omega}_k)$, C_1, $C_2(_{-1}\bar{\Omega}_k)$, and $C_3(_{-1}\bar{\Omega}_k)$) for $_1^0\bar{\Omega}_k$, the zeroth iterate approximation of $_1\bar{\Omega}_k$. Specifically

$$0 = \overbrace{[1 - E(_{-1}\bar{\Omega}_k)]^2}^{=:C_3(_{-1}\bar{\Omega}_k)} \cdot {}_1^0\bar{\Omega}_k(\varrho)^3$$
$$\underbrace{- 2\Pi_k[1 - E(_{-1}\bar{\Omega}_k)]}_{=:C_2(_{-1}\bar{\Omega}_k)} \cdot {}_1^0\bar{\Omega}_k(\varrho)^2$$
$$+ \underbrace{\Pi_k^2}_{=:C_1} \cdot {}_1^0\bar{\Omega}_k(\varrho) - C_0(_{-1}\bar{\Omega}_k), \tag{16}$$

which can be solved using Cardano's formula. We shall, however, skip doing this explicitly since it is standard and the formulas yield no new insight. In the range where $_1^0\bar{\Omega}_k$ is close to the "true" Ω_k, the above equation possesses one real and two complex conjugate roots, as this is the case for (13). Thus, by choosing the real root, a unique solution of the problem is obtained. The first five solutions of (16) are also visualized in Fig. 1 above.

To improve the level of accuracy further, we may now plug $_1^0\bar{\Omega}_k$ into E and C_0 to obtain an expression for $_1^1\bar{\Omega}_k$, etc., until we arrive at a self-consistent $_1^\infty\bar{\Omega}_k$ $(= {}_1\bar{\Omega}_k)$ which is the fixed point of the following itera-

tion scheme:

$$^{i+1}_1\bar{\Omega}_k = \mathscr{L}\{C_3(^{i-1}_1\bar{\Omega}_k) \cdot {}^i_1\bar{\Omega}_k^3 - C_2(^{i-1}_1\bar{\Omega}_k) \cdot {}^i_1\bar{\Omega}_k^2$$
$$+ C_1 \cdot {}^i_1\bar{\Omega}_k - C_0(^{i-1}_1\bar{\Omega}_k)\}. \tag{17}$$

Here $\mathscr{L}$ is the operator that corresponds to Cardano's formula and selects the real root. The indicated procedure converges after a few (typically 3) steps and corrects $^0_1\bar{\Omega}_k$ in most cases by about 1%. Thus the approximation that lead to (16) turns out to be well confirmed. Moreover, Fig. 1 depicts the fact that the step from $_{-1}\bar{\Omega}_k$ to $^0_1\bar{\Omega}_k$ already yields the major contribution in correcting the crude growth law towards the correct curves, that is, to remove the inherent constant error from (13). Unlike $_{-1}\bar{\Omega}_k$, the function $_1\bar{\Omega}_k$ converges towards Ω_k, since all errors die away like δ_k^2 or faster. Nevertheless, the speed of convergence is still rather slow because the δ_k themselves behave like $\varrho^{-1/3}$.

It is not hard to expand $z^{(\varphi)}$ nad $n^{(\varphi)}$ further, this, however, will introduce higher powers of $_i\bar{\Omega}_k$. Then the expressions analogous to (11) and (15) have to be treated numerically. This requires an numerical effort comparable to that for solving (7) itself. Thus we terminate our discussion of approximations at the present level, that yields results which are correct to order δ_k.

5. Discussions

Now that we have obtained the formal expressions for the limiting behavior of the curves Ω_k, let us interpret these results in terms of trajectories of the dynamical system. As was mentioned already, at parameter values from Ω_k a doubly touching trajectory comes into existence. That is, its entry point (s_k) and its first point of contact (b_k) emerge from the point x_w on W [7, 11]. Equivalently, as the final point x'_w of the critical curve φ collides with W, one specific touching trajectory gets a second point of contact (c_k) and becomes a doubly touching orbit. Thus, in this limiting situation, the mapping time τ_k^c is the time the system needs to move from one point on W to another one at the same line. $\Omega_k \tau_k^c$ is hence the phase difference of the two complex numbers $y_w := \eta(x_w) + i\xi_w(x_w)$ and $y'_w := \eta(x_w \cdot e^{-\varrho\tau_k^c}) + i\xi_w(x_w \cdot e^{-\varrho\tau_k^c})$, up to an integer multiple of 2π for every single one of the $k-1$ extra turns, which the orbit performs around the real eigenvector of the saddle-focus.

Let us now estimate the imaginary parts of the focal coordinates y_w and y'_w in a range of parameters that is of interest for our discussion, i.e., we assume $\varrho^2 \gg \Omega_k^2 \gg \varrho \gg 1$. Since $x_w \cdot e^{-\varrho\tau_k^c}$ decreases exponentially, $\mathscr{I}(y'_w)$ is essentially $x_0/2\Omega_k$ (more precisely, $0 < \mathscr{I}(y'_w) < x_0/2\Omega_k$), this means that it is bounded from below by zero and its upper bound goes to zero like to $\varrho^{-2/3}$. For the other imaginary part, $\mathscr{I}(y_w)$, we have to insert x_w into (3) and obtain with the above assumptions

$$\xi_w(x_w) = \frac{x_0}{2\Omega_k} \frac{\varrho(\varrho + 1 - \Omega_k^2)}{(\varrho + 1)^2 + \Omega_k^2} \approx \frac{-x_0}{2} \cdot \frac{\Omega_k}{\varrho}, \tag{18}$$

cf. (9) of [7]. It is now another easy consequence of (13) that the latter quantity approaches zero to order $\varrho^{-1/3}$. Thus both numbers y_w and y'_w tend (with different speeds) to be purely real and hence their difference in phase vanishes.

The argument presented above makes also clear, why the approximations on Ω_k get worse with growing values of k. To see how this happens, let us pick one arbitrarily fixed value of ϱ. Then, again from (13), the recursion formula $\Omega_{k+1}(\varrho) \approx \Omega_k(\varrho)\sqrt[3]{(k+1)/k}$ is obtained. Putting this result into (18), the sequence of values $\xi_w(x_w)$ is found to scale with the same factor as the Ω_k. Moreover, since $\tau_k^c \approx \Pi_k/\Omega_k$, the mapping times scale approximately like $\sqrt[3]{1 + 2/k + 1/k^2}$ and hence the imaginary part of y'_w grows and gets closer to its upper bound $x_0/(2\Omega_k)$ as k is increased. This visualizes that the sequence $\delta_k(\varrho)$ strictly increases with k. In consequence, the expansions approximating $z^{(\varphi)}$ and $n^{(\varphi)}$ become less accurate when we jump from one Ω_k to the next, for one fixed value of ϱ. The decrease in accuracy may be estimated by the "curves of constant phase shift" (δ_k = constant). To lowest order these curves turn out to be straight lines, as can be seen immediately from (12).

Let us finally mention that the growth law for the Ω_k could have been obtained also in a less constructive way. As was shown in [14], a "creation curve" Ψ_k^+ (which marks in three-region systems the first appearance of a doubly touching trajectory with points of contact on both separating planes) can be characterized by a collision of the critical curve φ with the return-transfer boundary Γ. Since this boundary is always from the halfplane S^+ (see [8]), the final point x'_w of φ crosses Γ prior to its reaching W. Hence, whenever the branching points (s'_k) of φ, in the instance of their coming into existence, are located to the left of Γ (a condition on ϱ for this behavior will be

presented in [15]), the curves Ω_k and Ψ_k^+ appear in an alternating manner as the frequency is increased (see Fig. 7 of [14]). The limiting growth law for the Ψ_k^+ was shown in [16] to be of the $\varrho^{2/3}$-type. Thus the Ω_k cannot grow with a different exponent. Otherwise these curves would have to intersect the creation curves Ψ_k^+. This, however, means that φ touches W and Γ simultaneously, contradicting the fact that the return-transfer boundary Γ is always located to the left of the line W.

To conclude, the charting of the canonical parameter space, as it was indicated earlier [7, 11], is now understood much better. The growth law for the curves Ω_k and its corrections allow us to detect the scaling behavior and other properties of frequencies and characteristic times. Hereby it is a remarkable fact that the canonical parameters, which determine the natural time scales of the dynamics, show no such "characteristic length". The curves Ω_k behave algebraically, and hence in the canonical parameter space all scales are found. The result presented is thus an important tool for a further development of the theory of piecewise-linear dynamical systems.

Acknowledgement

Appendix

Proof of the Lemma. A new doubly touching trajectory appears as the final point x'_W of φ reaches the line W. Let us first look for a lower and an upper bound for the phase difference of the two points of contact being $b_k = x_W$ and $c_k = x'_W$ [11]. Since both lie on the line W and have x-components less than x_0 their η-components are positive (cf. (2)). While the ξ-component of x'_W is always greater than zero ($x'_W < x_c$), the one of x_W has to be negative for all frequencies beyond $\sqrt{\varrho + 1}$ (a condition that is fulfilled for all Ω_k, see [7]), since in this case $x_W > x_c$ [7, 11]. Thus the phase difference between the two points is from the interval $(2\pi, 3\pi)$ – up to an integer multiple of 2π to be suppressed further on, like the index "k" of Ω and τ^c. In consequence, τ^c may vary from $2\pi/\Omega$ to $3\pi/\Omega$.

Let us now show that Ω grows slower than linearly with ϱ. First, if Ω stays bounded, (8) is fulfilled trivially, hence we may assume Ω to grow beyond any finite upper bound. Since simultaneous zeros of $z^{(\varphi)}$ and $n^{(\varphi)}$

are the conditions for the curves Ω in parameter space, (5) is the present object of investigation. In a region where $\Omega \gg 1$, the sine-term in (5a) can be neglected compared to $\Omega \cos(\Omega \tau^c)$. Hence the zeros of $z^{(\varphi)}$ in this case appear at a value that is close to the phase

$$\Omega(\varrho) \tau^2(\varrho) = \arccos[e^{-\tau^c(\varrho)}] = 2\pi + \delta(\varrho). \quad (A 1)$$

Instead of using this expression explicitly, we make use only of the fact that τ^c behaves like $1/\Omega$ and thus assume the zero of $z^{(\varphi)}$ to be shifted from 2π (or, equivalently, zero) by a phase δ, whose modulus tends toward zero for growing values of ϱ. Putting this into $n^{(\varphi)}$, neglecting "1's" compared to ϱ, and expanding to terms that contain δ to first order, yields for the zeros

$$0 = \Omega(\varrho) - e^{\varrho\tau^c(\varrho)}[\Omega(\varrho) - \varrho\delta(\varrho)]. \quad (A 2)$$

Implications:

(1) Let us assume that the function Ω grows faster than linearly. Then the right-hand side of (A 2) may be written as

$$H_{L+} := \Omega(\varrho)\left\{1 - \exp\left[\frac{\varrho\sigma}{\Omega(\varrho)}\right]\left(1 - \frac{\varrho\delta(\varrho)}{\Omega(\varrho)}\right)\right\}, \quad (A 3)$$

where σ is the true phase that has to be from the interval $(2\pi, 3\pi)$. Since ϱ/Ω is assumed to approach zero, we expand H_{L+} in terms of ϱ/Ω and obtain

$$H_{L+} = \varrho\left[\delta(\varrho) - \sigma + \mathcal{O}\left(\left|\frac{\varrho}{\Omega(\varrho)}\right|\right)\right]. \quad (A 4)$$

This shows that (for $\varrho \to \infty$) H_{L+} is dominated by the $(\varrho\sigma)$-term. It thus does not vanish, as is required for $n^{(\varphi)}(\tau^c(\varrho); \varrho, \Omega(\varrho))$. Hence our first assumption yields a contradiction to the prerequisite of the *Lemma*.

(2) Next, let us assume that $\varrho/\Omega \to \mu$ as ϱ grows, i.e., Ω increases linearly with ϱ. In this case the right-hand side of (A 2) can, for great values of ϱ, be approximated by

$$H_L := \mu\varrho\left\{1 - e^{\sigma\mu}[1 - \mu\delta(\varrho)]\right\}. \quad (A 5)$$

For $\varrho \to \infty$ this function behaves like $\varrho\mu(1 - e^{\sigma\mu})$ and hence (since $\varrho\mu, \sigma\mu > 0$) gows linearly towards $-\infty$ instead of vanishing. This again contradicts the required zero of $n^{(\varphi)}$ along the curve Ω.

Thus we have proved that Ω indeed increases slower than linearly and both expressions ϱ/Ω and $\varrho\tau^c = \varrho\sigma/\Omega$ grow beyond any given upper bound.

q.e.d.

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$$\lim_{x \nearrow s} \bar{P} \circ \tilde{P} \circ \bar{P}(x) = \lim_{x \searrow s} \bar{P}(x)$$

and (10a) should read

$$d_3(\tau) := \varrho\,\omega + e^\tau \{(\omega^2 + \varrho + 1)\sin \omega\tau - \varrho\,\omega \cos \omega\tau\}$$
$$- e^{(\varrho + 1)\tau} \{(\omega^2 + \varrho + 1)\sin \omega\tau + \varrho\,\omega \cos \omega\tau\}$$
$$+ e^{(\varrho + 2)\tau} \varrho\,\omega .$$

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